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Technical Report

JSR-80-22

September 1980

**THEORY OF THE TRANSVERSE
GRADIENT WIGGLER**

By: N. Kroll
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M. Rosenbluth

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20 ABSTRACT (Continued)

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Abstract

Three operating regimes are identified, characterized as low, medium, and high optical intensity regimes. The first two are Raman regimes in which the optical wave and transverse betatron oscillations grow simultaneously. They differ from one another only in the nature of the saturation mechanism. The third or high intensity regime is a trapped particle regime physically very similar to the regime of high extraction variable parameter wigglers. A comparison is made between the two.

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THEORY OF THE TRANSVERSE GRADIENT WIGGLER

I. Introduction

J. Madey and co-workers have proposed the use of an FEL with a transverse field gradient to prevent the energy spreading and detuning which saturates the ordinary FEL. The basic idea is that the transverse field gradient acts as a spectrometer so that as an electron's energy changes due to interaction with radiation it is focused to a different transverse position. One then arranges the transverse field gradient so that the resonance condition $k_s = \frac{2\gamma^2(x)}{1 + a_w^2(x)} k_w$ remains satisfied at all transverse positions, x . Here k_s is the radiation wave number, k_w the wiggler wave number, and $a_w = \frac{eB_w}{mc^2 k_w}$, the vector potential of the wiggler magnet field. a_w is usually close to unity in value. We will concentrate our attention on the possibility of obtaining high extraction from such a magnet.

Madey¹ has proposed a wiggler structure characterized in the $y = 0$ plane by:

$$\hat{B}(z, x) = \hat{e}_y \left[\sqrt{2} B(x) \cos k_w z + B_c(x) \right]$$

$$B(x) = B_0 \left[\left(1 + \frac{x}{\eta} \right)^2 + \frac{x}{\eta} \frac{2 + x/\eta}{a_w^2} \right]^{1/2}$$

and

$$B_c(x) = \frac{1}{\gamma_0} \frac{e}{mc^2 k_w^2} \frac{B(x)B'(x)}{1 + \frac{x}{\eta}}$$

Thus the field gradients are characterized by η^{-1} . This choice preserves the transverse independence of resonance. The equations of motion for an electron orbiting in the $x - z$ plane [after averaging over the fast (k_w^{-1}) motion] have been given by Madey as:

$$\frac{d^2x}{dz^2} = -k_\beta^2 (x - n\delta)(1 + \delta)^{-2} \quad (1)$$

$$\frac{d\delta}{dz} = \frac{e^2 |E|}{mc k_w \gamma_0^2} J \sin \psi(z) \left[1 + \frac{x}{\eta} \left(1 + \frac{1}{2} \frac{1}{a_w} \right) - \frac{2\delta}{a_w} \right] \quad (2)$$

$$\frac{d\psi}{dz} = q + 2k_w \frac{\left(1 + \frac{1}{2} (\delta + x/\eta) \right)}{(1 + \delta)^2} \left(\delta - \frac{x}{\eta} \right) - \frac{\gamma^2}{1 + \frac{1}{a_w^2}} k_w \left(\frac{dx}{dz} \right)^2 \quad (3)$$

Here $|E|$ is the electric field of the optical wave, $\delta = \frac{\gamma}{\gamma_0} - 1$ with γ_0 being the design initial energy for a particle which enters at $x = 0$, and ψ is the optical phase $\psi = (k_s + k_w)z - \omega_s \int \frac{dz}{v_z}$. q , the optical phase slip, is, of course, a function of frequency. The design has been chosen such that v_z , and hence q is independent of electron energy so long as the betatron focussing keeps the electron at $x = n\delta$. Further, we have

$$k_{\beta}^2 = \frac{1}{n^2} \frac{(1 + a_w^2)}{\gamma^2} \ll k_w^2 \quad (4)$$

and J is a constant close to unity.

The underlined terms in equations (1) - (3) are small. Moreover, there is some disagreement among Madey's papers, and further with recent work of Morton, as to their exact coefficients which depend on details of design. However, all workers are agreed as to their approximate magnitude and form and we shall see that the nature of the solution is insensitive to the details of these small non-linear effects. We will first consider the linearized problem, neglecting the underlined terms.

First, however, it is convenient to introduce dimensionless variables. Let

$$\begin{aligned} z &\rightarrow \frac{z}{k_{\beta}} \\ x &\rightarrow \frac{k_{\beta}}{2k_w} \eta x \\ \delta &\rightarrow \frac{k_{\beta}}{2k_w} \delta \\ q &\rightarrow k_{\beta} q \end{aligned} \quad (5)$$

The equations transform to (keeping terms of order $\frac{k_{\beta}}{k_w}$)

$$\ddot{x} = -(x - \delta) \left(1 - \frac{k_{\beta}}{k_w} \delta\right) \quad (6)$$

$$\dot{\delta} = \varepsilon \sin \psi \left[1 + \frac{k_{\beta}}{2k_w} x (1 + a_w^{-2}) - \delta \right] \quad (7)$$

$$\dot{\psi} = q - (x - \delta) \left[1 + \frac{k_{\beta}}{4k_w} (x - 3\delta) \right] - \frac{k_{\beta}}{4k_w} x^2 \quad (8)$$

It may be seen that the non-linear terms are all formally of order

$$\frac{k_{\beta}}{k_w} \ll 1 \quad . \quad \text{Further } \varepsilon \equiv \frac{e^2 |E| B_0}{m c^4 k_{\beta}^2 \gamma_0^2} J = 4 \frac{a_s a_w}{1 + a_w^2} \frac{k_w^2}{k_{\beta}^2} J = \frac{\Omega^2}{k_{\beta}^2} \quad \text{where } \Omega \text{ is the}$$

synchrotron frequency for oscillation in the ponderomotive potential

wells. In Sections II and IV we discuss regimes in which

$$\varepsilon = (\Omega/k_{\beta})^2 \ll 1 \quad . \quad \text{The case } \varepsilon \gg 1 \text{ is discussed in Section V.}$$

II. Linearized Solution for $\epsilon \ll 1$.

The linearized forms of equations (6), (7), and (8) are:

$$\ddot{x} = -(x - \delta) \quad (9)$$

$$\delta = \epsilon \sin \psi \quad (10)$$

$$\dot{\psi} = q - (x - \delta) = q + \ddot{x} \quad (11)$$

Equation (11) may be integrated immediately to yield

$$\psi = \psi_0 + qz + \dot{x} \quad (12)$$

where we have absorbed the initial value of \dot{x} into the initial (random) optical phase ψ_0 . Differentiating equation (9) then yields the desired equation for $X \equiv \dot{x}$.

$$\ddot{X} = -X + \epsilon \sin (\psi_0 + qz + X) \quad (13)$$

Equation (13) is evidently of the form of a driven harmonic oscillator. We are interested in achieving a high degree of excitation. Thus the interesting regime is $q = 1$, a nearly resonant oscillator. We look for a solution of the form

$$X = f(z) \sin [qz + \phi(z)] \quad (14)$$

where f and ϕ are slowly varying in z .

Thus

$$\begin{aligned}\ddot{X} &= 2(q + \dot{\phi}) \dot{f} \cos(qz + \phi) + \left[\ddot{f} - (q + \dot{\phi})^2 f \right] \sin(qz + \phi) \\ &\approx 2\dot{f} q \cos(qz + \phi) - f(q^2 + 2q\dot{\phi}) \sin(qz + \phi)\end{aligned}\quad (15)$$

In deriving equation (15) we have made use of the assumption that f and ϕ are slowly varying.

Further, using the Bessel function expansion

$$e^{if \sin \lambda} = \sum_{n=-\infty}^{\infty} e^{in \lambda} J_n(f)$$

we have

$$\sin(\psi_0 + qz + X) = \sum_{n=-\infty}^{\infty} J_{n-1}(f) \sin \left[n(qz + \phi) - (\phi - \psi_0) \right]$$

Here we need only the secular term ($n = 0$) and the resonant terms $n = \pm 1$.

Thus

$$\begin{aligned}
\sin (\psi_0 + qz + X) &= J_1(f) \sin (\phi - \psi_0) \\
&+ 2J_1'(f) \sin (qz + \phi) \cos (\phi - \psi_0) \\
&- 2 \frac{J_1(f)}{f} \cos (qz + \phi) \sin (\phi - \psi_0)
\end{aligned}
\tag{16}$$

We are now in a position to substitute equations (15) and (16) into (13) and equate coefficients of $\sin (qz + \phi)$ and $\cos (qz + \phi)$. First, however, we specialize to the near resonant case by defining

$$1 - q^2 = q\lambda \tag{17}$$

and setting $q \equiv 1$ elsewhere in the equations. From equation (13), we then have

$$\ddot{f} = -\epsilon \frac{J_1(f)}{f} \sin (\phi - \psi_0) \tag{18}$$

and

$$f\dot{\phi} = \frac{\lambda f}{2} - \epsilon J_1'(f) \cos (\phi - \psi_0) \tag{19}$$

Further, we may determine the energy change from equation (10) and the secular term in equation (16).

$$\dot{\delta} = \epsilon J_1(f) \sin(\phi - \psi_0) = -f\dot{f}$$

so that

$$\delta = \frac{f_0^2 - f^2}{2}, \quad (20)$$

with $f_0 \ll 1$ the initial value of f .

Note that the non-secular terms in equation (16) would lead to oscillatory terms in δ which are $O(\epsilon) \ll 1$, indicating small energy spread.

Dividing equation (19) by equation (18) we have

$$\frac{d \cos(\phi - \psi_0)}{df} = \frac{\lambda f}{2\epsilon J_1} - \frac{J_1'}{J_1} \cos(\phi - \psi_0)$$

with solution

$$\cos(\phi - \psi_0) = \frac{1}{J_1} \left[C + \frac{\lambda}{4\epsilon} f^2 \right]. \quad (22)$$

We will be concerned with low initial emittances $f_0 \ll 1$ in which case $C = 0$. In this case we see that near

$f = 0$, $\cos(\phi - \psi_0) = 0$, while from equation (19) it is clear the stable point is $\phi - \psi_0 = -\frac{\pi}{2}$. Hence initially $\sin(\phi - \psi_0) = -1$ and we see from equation (18) that f initially increases. We may now write down the solution to equation (18)

$$z = \frac{1}{\epsilon} \int_{f_0}^f \frac{f' df'}{\left[J_1^2(f') - \left(\frac{\lambda f'^2}{4\epsilon} \right)^2 \right]^{1/2}} \quad (23)$$

Equations (20) and (23) represent the solution to the problem. Saturation will occur for f_{\max} given by

$$J_1(f_{\max}) = \frac{\lambda f_{\max}^2}{4\epsilon} .$$

Since ϵ is by assumption small we find for particles off resonance ($\lambda \neq 0$) that $f_{\max} = \frac{2\epsilon}{\lambda}$ and $-\delta_{\max} = \frac{2\epsilon^2}{\lambda}$. For resonant particles and an infinitely long wiggler, growth will stop at $J_1(f) = 0$, i.e., $f = 3.8$. Here $\delta_{\max} = -\frac{(3.8)^2}{2}$. Thus the maximum extraction which can be obtained by such a gain-expanded wiggler in the small ϵ regime is (using equation 5):

$$-\frac{\delta\gamma}{\gamma} = \frac{(3.8)^2}{4} \frac{k_\beta}{k_w} . \quad (24)$$

By comparison, the maximum extraction from a simple wiggler is Ω/k_w , which is smaller. (The saturation length for a simple wiggler is $\frac{\pi}{\Omega}$.)

We may obtain an explicit evaluation of equation (23) in the small signal limit since $J_1(f) = \frac{1}{2}f$ for f small. Then from (23) and (20) we have

$$-\frac{\delta\gamma}{\gamma} = \frac{k_{\beta}}{4k_w} f^2 = \frac{k_{\beta}}{k_w} \frac{\epsilon^2}{2} \sin^2 \frac{\lambda}{4} z \quad (25)$$

A reasonable approximation to the departure from the small signal limit for $\lambda^2 \gg 3\epsilon^2$ is obtained by approximating $J_1^2(f) = \frac{1}{4} f^2 (1 - \frac{1}{4} f^2)$, which yields:

$$-\frac{\delta\gamma}{\gamma} = \frac{k_{\beta}}{k_w} \frac{\epsilon^2}{\lambda^2 + \epsilon^2} \sin^2 \frac{(\lambda^2 + \epsilon^2)^{1/2}}{4} z \quad (25')$$

The shape of the resonant ($\lambda = 0$) extraction curve, obtained by numerical evaluation of equation (23) is shown in Figure 1. An approximate saturation length can be defined by

$$L_{\text{sat}} = \frac{8}{k_{\beta}\epsilon} = \frac{8k_{\beta}}{\Omega^2} \quad (26)$$

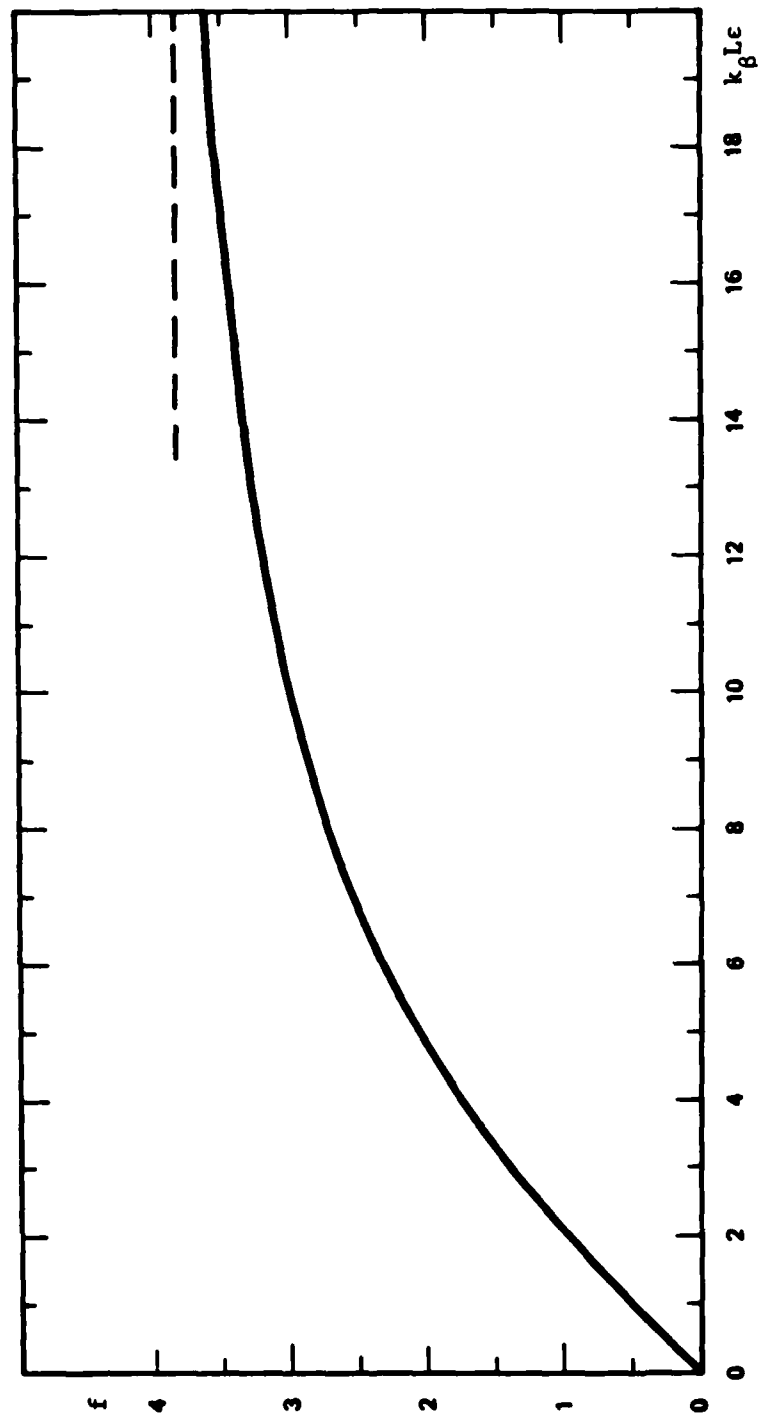


FIGURE 1 Oscillation amplitude f at resonance, shown as a function of interaction length for the intermediate intensity regime.

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III. Non-Ideal Effects

We may now discuss the sensitivity and modification of our solution by various perturbing effects.

(1) Horizontal Emittance

We have assumed that at $z = 0$, $x = \eta\delta$ and $\dot{x} = 0$. If this is not the case then the constant C of equation (22) may no longer be set to zero. A complete analysis in the small signal limit at resonance is straightforward and shows that, after phase averaging, $\langle\delta\rangle$ is unaffected by the value of f_0 even when $f_0^2 > f^2 - f_0^2$, so long as $f_0 < 1$. It is also clear, however, that as the initial f_0 increases significantly beyond unity, the saturated limit of δ decreases. We may estimate that the horizontal emittance E_h is related to f_0 by

$$E_h = x_0 x_0' = k_\beta x_0^2 = k_\beta \left(\frac{k_\beta \eta}{k_w} f_0 \right)^2$$

so that the condition on maximum allowed horizontal emittance is

$$f_0^2 = 2E_h \frac{k_w}{k_\beta} < 1 \quad (27)$$

(2) Vertical Emittance

For a simple 2-D gain expander of the type proposed by Madey the vertical and horizontal oscillations are decoupled. We have not investigated the possibility of a more complex 3-D structure. For the decoupled magnet it may be seen that the vertical oscillation frequency

$k_{\beta v}$ is of the same order of magnitude as k_{β} . The variation in flutter motion with y makes a comparable contribution to Δv_z . The exact relations depend upon details of magnet design. As a simple parametrization we write

$$\frac{E}{v} \sim \frac{y'^2}{k_{\beta}'} \sim \frac{2\Delta v_z}{k_{\beta}'}$$

where $k_{\beta} \sim k_{\beta}'$. Requiring that the phase slip be less than π and using equation (26) to estimate L we find

$$k_s E_v \left(\frac{k_{\beta}'}{\Omega} \right)^2 < \frac{\pi}{4}, \quad (28)$$

(3) Diffraction

The lengths implied by equation (26) may also pose difficulties in passing a diffraction limited beam through the magnet. Using the criterion that the optical beam intensity for a Gaussian beam should be down by e^4 at the magnet aperture from its central value, and using equation (26) to estimate beam length we find a restriction on beam radius

$$r_b^2 > 32 \frac{k_{\beta}}{\Omega^2 k_s} \quad (29)$$

Since geometrical considerations may also require $r_b \sim \eta$ equation (29) may provide considerable restraint on magnet design.

Equations (26) - (29) and the definitions:

$$k_s = \frac{2\gamma^2}{1 + a_w^2} k_w \quad (30)$$

$$k_\beta = \frac{1}{\eta} \frac{\sqrt{1 + a_w^2}}{\gamma} \quad (31)$$

$$\Omega^2 = \frac{4a_s a_w}{1 + a_w^2} k_w^2 \quad (32)$$

$$a_w = \frac{eR_w}{mc^2 k_w} \quad (33)$$

provide the basic input for specifying the parameters of a medium-extraction single-pass gain-expanded magnet.

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IV. Effect of the Omitted Non-Linear Terms

We turn now to a consideration of the modifications introduced by the underlined terms in equations (6) - (8). First, however, we briefly consider a related problem, the effect of using wiggler magnets whose properties a_w , k_w , etc., vary with z . It seems clear that the principal effect, in view of the sharp resonance behavior exhibited by equation (25), will be to make $\lambda(z)$ variable. Thus if we modify our basic ansatz (14) to assuming a solution of the form

$X = f(z) \sin \left(z - \frac{1}{2} \int \lambda dz + \phi \right)$ we may reduce the problem to a solution of equations (18) and (19) with λ replaced by $\lambda(z)$. For general $\lambda(z)$ these may be reduced to a non-linear 2nd order differential equation for f . However, it is clear that the saturation limit of equation (24) will not be exceeded.

Let us now return to the non-linear problem and consider equations (6), (7), and (8). Working to order x^2 , δ^2 , $x\delta$, etc., and making use of the fact that δ is slowly varying to $O(\epsilon)$ while x has a large oscillatory piece, we may rewrite equation (8) as

$$\begin{aligned} \ddot{\psi} &= q + \ddot{x} \left[1 + \frac{k_\beta}{4k_w} (x - \delta + 2\delta) \right] - \frac{k_\beta}{4k_w} \dot{x}^2 \\ &= q + x \left[1 + \frac{k_\beta}{2k_w} \delta \right] - \frac{k_\beta}{4k_w} (\dot{x}^2 + \ddot{x}^2) \end{aligned}$$

so that

$$\begin{aligned}\psi &= qz + \psi_0 - \frac{k_\beta}{4k_w} \int (\dot{x}^2 + \ddot{x}^2) dz + \dot{x} \left[1 + \frac{k_\beta}{2k_w} \delta \right] \\ &= qz + \psi_0 - \frac{k_\beta}{4k_w} \int (\dot{x}^2 + \ddot{x}^2) dz + X\end{aligned}\quad (34)$$

and from equation (6)

$$\ddot{X} = -\left(1 - \frac{k_\beta}{k_w} \delta\right) X + \dot{\delta} \left(1 - \frac{k_\beta}{2k_w} \delta\right) \quad (35)$$

Again we make the ansatz

$$X = f \sin \left[\psi_0 + qz - \frac{k_\beta}{4k_w} \int (\dot{x}^2 + \ddot{x}^2) dz + \phi \right]$$

and keep only those terms which affect the resonance. Thus it is adequate to use $\dot{\delta} = \epsilon \sin \psi$, neglecting the order $\frac{k_\beta}{k_w}$ modifications in ϵ . Equations (18) and (20) are then unaltered while equation (19) becomes [recalling that $\dot{X} = f \cos(\)$; $\ddot{X} = -f \sin(\)$, $\delta = -f^2/2$]

$$\dot{\phi} = \frac{\lambda + \frac{k_\beta}{k_w} f^2}{2} - \frac{\epsilon J_1'(f)}{f} \cos(\phi - \psi_0) \quad (36)$$

As we have done in the linear theory we may integrate equations (18) and (36) to obtain

$$\cos(\phi - \psi_0) = \frac{1}{J_1} \left[\frac{\lambda}{4\epsilon} f^2 + \frac{k_\beta}{k_w} \frac{f^4}{8\epsilon} \right] \quad (37)$$

and

$$z = \frac{1}{\epsilon} \int_0^f \frac{f df}{\left[J_1^2 - \left(\frac{\lambda}{4\epsilon} f^2 + \frac{k_\beta}{k_w} f^4 / 8\epsilon \right)^2 \right]^{1/2}}$$

Recalling that the resonance width is determined by $\frac{\lambda z}{4} \lesssim \frac{\pi}{2}$,

we see that the non-linear effect will be unimportant when $\frac{k_\beta}{k_w} \lesssim \frac{4\pi}{k L f^2}$.

Taking $f^2 = 8$ as a typical saturation value and using (26), this criterion

may also be written $\frac{k_\beta}{k_w} < \frac{4\pi}{64} \epsilon$. When this criterion is well satisfied

the saturation level of equation (24) becomes accessible if λ is given an appropriately chosen small negative value.

We see that non-linear effects become important when $\frac{k_\beta}{\epsilon k_w}$ is large enough. For the resonant electron, $\lambda = 0$, peak extraction will then occur for

$$\frac{f^4 k_\beta}{8\epsilon k_w} = J_1(f) = f/2$$

or

$$-\frac{\delta\gamma}{\gamma} = \frac{k_\beta}{4k_w} f^2 = \left(\frac{k_\beta}{4k_w} \right)^{1/3} \epsilon^{2/3} \quad (38)$$

at

$$z = \frac{4}{\epsilon} \left(\frac{\epsilon k_w}{k_\beta} \right)^{1/3} \frac{\Gamma^2(7/6)}{\Gamma(4/3)} = 3.85 \left(k_w / k_\beta \epsilon^2 \right)^{1/3}$$

By detuning, i.e., choosing a slightly negative λ , a factor of about 2.5 improvement can be obtained in extraction, while of course a carefully chosen variable pitch $\lambda(z)$ could cancel the non-linear frequency shift and recover the saturated efficiency of equation (24). As discussed earlier, different magnet designs could also result in slightly different non-linear terms thereby yielding a different numerical coefficient in equation (38). However, it is very difficult to see how a design of this type could improve on the extraction efficiency given by equation (24). We have considered in this note only designs for $q = 1$ as proposed for small ϵ high single pass extraction. However, the techniques employed herein may also prove of use in analyzing the behavior of functionally canceling designs proposed for storage ring adaption. Finally, we note that operation at $q = 0$ could also give comparable extraction if a pre-bunched beam (Klystron) were used.

We have integrated equations (6) - (8) numerically for the case $\epsilon = .05$, $\frac{k_\beta}{k_w} = .2$ for which equation (38) predicts $(-\frac{\delta\gamma}{\gamma})_{\max} = .050$. Numerically we obtained $(-\frac{\delta\gamma}{\gamma})_{\max} = .051$.

V. The High Intensity Regime

We recall from the theory of the uniform wiggler that the parameter Ω corresponds to the small amplitude oscillation frequency in the ponderomotive potential wells, and may be referred to as the optical synchrotron frequency. In our previous discussion we have assumed this frequency to be small compared to k_β , the betatron frequency, here induced by the transverse gradient. The case in which $\Omega \sim k_\beta$ can be expected to be complicated as it is likely to lead to coupling between the two oscillation modes. However, the case $\Omega \gg k_\beta$ again leads to behavior which can be discussed in a simple way.

We first differentiate equation (11) to obtain $\ddot{\psi} = -(\dot{x} - \dot{\delta})$, and using equations (10) and (12) we find

$$\ddot{\psi} = \epsilon \sin \psi + \psi_0 - \psi + qz \quad (39)$$

To relate the discussion to that given in Ref. (2) we shift the phase in (39) of both ψ and ψ_0 by π . That is, we write $\psi = \phi + \pi$, which yields

$$\ddot{\phi} = -\epsilon \sin \phi + \phi_0 + qz \quad (40)$$

Since $\epsilon = \frac{\Omega^2}{k_\beta^2}$, we are here interested in the region $\epsilon \gg 1$. For $qz \ll \epsilon$, the equation is approximated by $\ddot{\phi} = -\epsilon \sin \phi$. Since $\dot{\phi}(0) = q$ (for $x = \delta$), those electrons with $-\pi + \frac{q}{\sqrt{\epsilon}} < \phi_0 < \pi - \frac{q}{\sqrt{\epsilon}}$

are trapped in oscillatory orbits. That is, all but a fraction $q/\pi\sqrt{\epsilon}$ are trapped, and we assume this quantity to be very small. For the trapped orbits $|\phi|$, $|\phi_0| < \pi$ and we can therefore consistently treat these terms as small compared to ϵ . The term qz , of course, grows with z and we shall show below that it leads to a progressive detrapping of particles, but it also leads to a reduction of particle energy for those which remain trapped. Appealing to the theory of the variable parameter wiggler discussed in Ref. 2, we shall assume that the main contribution to the averaged energy exchange comes from the trapped particles, and neglect any contribution from particles which have become detrapped. Thus for trapped particles equation (40) may be written

$$\ddot{\phi} = -\epsilon \sin \phi + qz = -\frac{dF}{d\phi} \quad (41)$$

where

$$F(\phi) = -\epsilon (\cos \phi + \phi \sin \phi_r) \quad (42)$$

with

$$\sin \phi_r = \frac{qz}{\epsilon} \quad (43)$$

Equation (41) describes motion in the potential $F(\phi)$, which has the general form of Figure 2.1 of Ref. 2. The trapped particles execute oscillatory motion about the minimum, which occurs at $\phi = \phi_r$, and for

the trapped particles $\langle \sin \phi \rangle$, which represents the average over this motion, is equal to $\sin \phi_r$. This statement is rigorously true only for z independent ϕ_r , but the same result holds approximately when we may regard ϕ_r as varying adiabatically. Again assuming adiabatic variation, the phase area of any trapped orbit remains constant, but the maximum phase area available for trapped orbits shrinks as ϕ_r increases and vanishes as ϕ_r reaches $\frac{\pi}{2}$. There is thus a continual detrapping of particles as z increases, and when $qz = \epsilon$ all particles are detrapped. Since there is then no further average energy exchange, the wiggler should be terminated when, or before, this point is reached.

On the basis of the above physical description we write

$$\begin{aligned} \dot{\langle \delta \rangle} &= -\epsilon \sin \psi_r j(\sin \psi_r) \\ &= -qz j\left(\frac{qz}{\epsilon}\right) \end{aligned} \quad (44)$$

where $j(\sin \phi_r)$ is the area in phase space enclosed by the last trapped orbit relative to that at $\phi_r = 0$. It is given explicitly by

$\alpha(\psi_r) = \alpha(\sin^{-1} \frac{qz}{\epsilon})$ defined by equation 2.60 in Ref. (2). It follows that $j(0)$ is unity, and if $\frac{qz}{\epsilon} \ll 1$, equation (44) yields the simple result

$$\langle \delta \rangle = -\frac{1}{2} qz_M^2$$

where $z_M = k_\beta L$. Reintroducing the original variables in accordance with equation (5), we have

$$-\frac{\langle \delta \gamma \rangle}{\gamma} = \frac{q(k_\beta L)^2}{4k_w} \quad (45)$$

for $qL \ll \epsilon$. The function $j(\frac{qL}{\epsilon})$ is obtained by carrying out the integral of equation 2.60 (Ref. 2) numerically. A second numerical integration of equation (44) is then required to obtain $\langle \delta \rangle$ when $\frac{qL}{\epsilon}$ is not small. We write the result as

$$-\frac{\langle \delta \gamma \rangle}{\gamma} = \frac{q(k_\beta L)^2}{4k_w} G\left(\frac{qL}{\epsilon}\right) \quad (46)$$

where G is plotted in Figure 2. We note that G decreases approximately linearly from one to .22 as its argument varies between zero and one. As noted before, regarded as a function of L , $\langle \delta \rangle$ reaches its maximum value at $qL = \epsilon$, yielding

$$\left(\frac{-\langle \delta \gamma \rangle}{\gamma}\right)_L = \frac{\Omega^4}{4k_w q k_\beta^2} (.22) \quad (47)$$

Equation (46) can also be conveniently written in the form

$$-\frac{\langle \delta \gamma \rangle}{\gamma} = \frac{\Omega^2 L}{4k_w} \frac{qL}{\epsilon} G\left(\frac{qL}{\epsilon}\right) \quad (48)$$

The product $\frac{qL}{\epsilon} G\left(\frac{qL}{\epsilon}\right)$ is also plotted in Figure 2. It is seen to have maximum value of .27 at $\frac{qL}{\epsilon} \approx .65$. Thus we may write

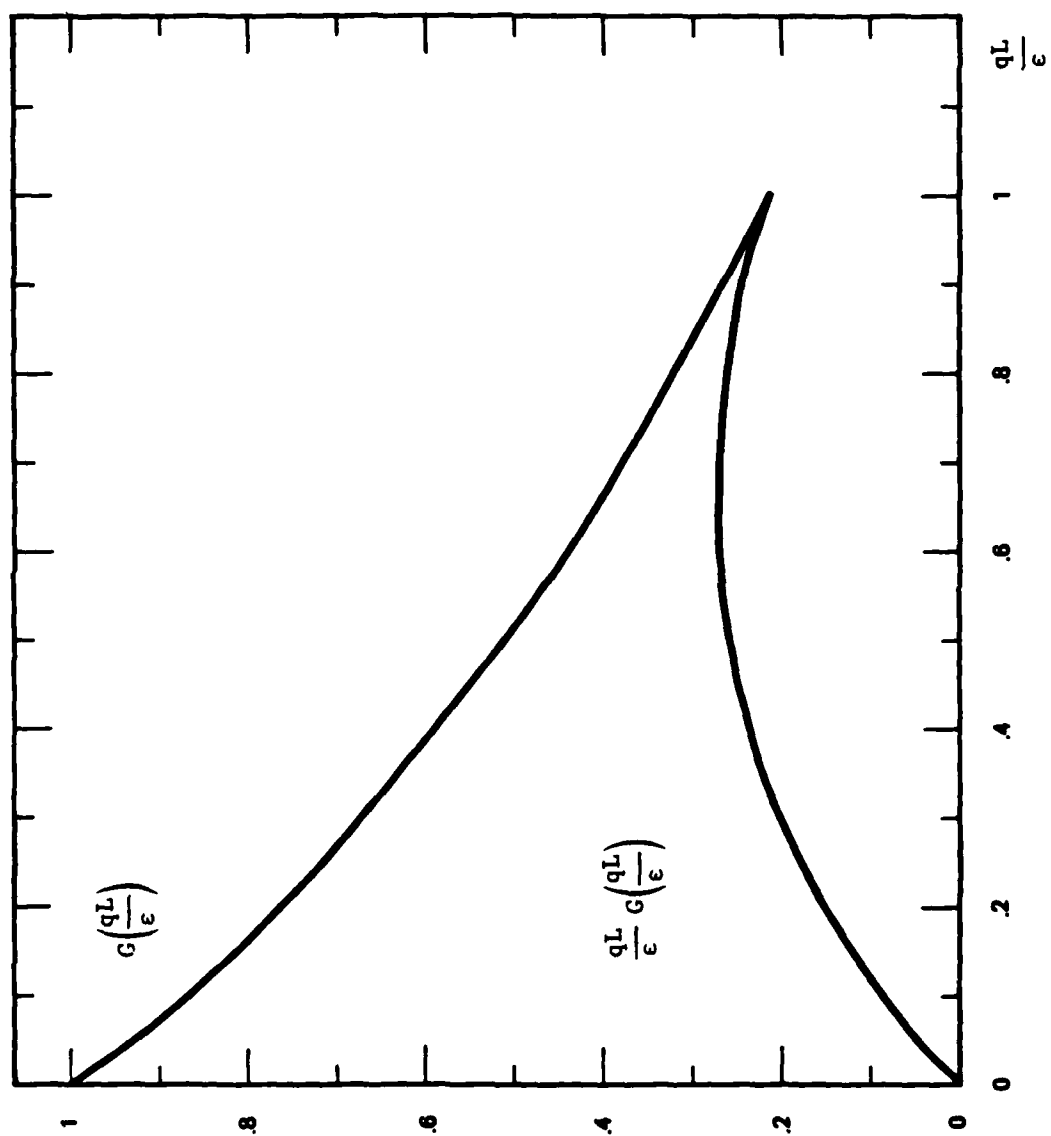


FIGURE 2 The Extraction Saturation Function for the High Intensity Regime

(for the optimum q)

$$\left(-\frac{\langle \delta \gamma \rangle}{\gamma}\right)_{\max} = \frac{\Omega^2 L}{4k_w} \quad (.27) \quad (48)$$

The horizontal and vertical emittance requirements which are applicable to this regime differ from those derived in III. In the case of vertical emittance we again have

$$E_v = \frac{2\Delta v_z}{k'_\beta}$$

but here we require that Δv_z be sufficiently small to permit trapping, i.e., as shown in (Ref. 2)

$$\Delta v_z < \frac{1 + a_w^2}{\gamma^2} \frac{\delta \gamma_m}{\gamma} = \frac{1 + a_w^2}{\gamma^2} \frac{\Omega}{k_w}$$

so that

$$E_v < \frac{2(1 + a_w^2)}{\gamma^2 k_w} \sqrt{\epsilon} \frac{k_\beta}{k'_\beta} \quad (50)$$

In the case of horizontal emittance it is necessary to take the coupling of the transverse motion to the optical synchrotron oscillations into account. When $\dot{x}(0)$ does not vanish, equation (40) must be written

$$\ddot{\phi} = -\epsilon \sin \phi + \phi_0 - \phi + [qz - \dot{x}(0)] \quad (51)$$

The condition for the existence of a trapping potential at $z = 0$ is

$\dot{x}(0) < \epsilon$. Furthermore, $\dot{\phi}(0) = q - (x - \delta) = -(x - \delta)$. Trapping for $\dot{x}(0) = 0$ requires $(x - \delta)_0 < 2\sqrt{\epsilon}$. In order to take the two effects into account simultaneously we estimate (for a properly matched beam)

$$(x - \delta)_0 \dot{x}(0) < \epsilon^{3/2}$$

Using equation (5) to regain dimensional variables and equation (4) to eliminate η , we obtain

$$E_h = [(x - \delta)_0 \dot{x}(0)]_{\text{dimensional}} < \frac{k_\beta(1 + a_w^2)}{4k_w^2 \gamma^2} \epsilon^{3/2} = \frac{k_\beta}{2k_w k_s} \epsilon^{3/2} \quad (52)$$

In conclusion we note that these crude approximations may be expected to hold when $\Omega L \gg 1$ but $\frac{\Omega}{k_w} \ll 1$ and $\delta \ll 1$ also. As seen in Section IV, the principal effect of the non-linear terms is to shift the betatron frequency as the betatron amplitude increases. For the regime discussed in this section, the resonance condition $q = k_\beta$ plays no role, the excitation of betatron oscillations is small, and we expect the non-linear terms to be less important than other omissions in the analysis.

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VI. Summary and Discussion

In the preceding sections we have identified three distinct operating regimes for the transverse gradient wiggler (TCW), which we characterize as low intensity, medium intensity, and high intensity regimes. The parameters and properties of these regimes are summarized in Table I. In the interest of simplicity and in recognition of the lack of numerical precision in some of our arguments we have simplified numerical coefficients. The more detailed description is given in the previous text.

The low and medium intensity regimes may be thought of as stimulated Raman effect regimes, in which a quantum of betatron oscillation and an optical quantum are simultaneously emitted, with the energy being supplied by the reduction of the amplitude in the transverse flutter motion. In more classical terms, the electron in its longitudinal rest frame sees a ponderomotive potential which oscillates at the transverse betatron frequency. This induces oscillations in γ which, on account of the transverse field gradients, results in a transverse driving force at the betatron frequency. The combination of the flutter motion and induced betatron motion has sum and difference frequency components, and as is typical of Raman processes, only the difference frequency component leads to amplification. The frequency of the amplified wave in the laboratory frame is then found by Doppler shifting the frequency determined in the electron's longitudinal rest frame. As the electron loses energy to the optical wave, the center of betatron oscillation shifts transversely, so that the electron's average transverse position shifts to a position of

TABLE I

	Low Intensity Regime	Medium Intensity Regime	High Intensity Regime
definition of regime	$0 < \frac{\Omega}{k_{\beta}} \ll 3 \left(\frac{k_{\beta}}{k_w} \right)^{1/2}$	$3 \left(\frac{k_{\beta}}{k_w} \right)^{1/2} \ll \frac{\Omega}{k_{\beta}} \ll 1$	$\frac{\Omega}{k_{\beta}} \gg 1$
$\frac{\Delta Y}{Y}$ ($L \ll L_{sat}$)	$\frac{k_{\beta}}{16k_w} \frac{\Omega^4}{k_{\beta}^2} L^2$	$\frac{k_{\beta}}{16k_w} \frac{\Omega^4}{k_{\beta}^2} L^2$	$\frac{k_{\beta}}{4k_w} q k_{\beta} L^2$
$\left(\frac{\Delta Y}{Y} \right)_{max}$	$\left(\frac{k_{\beta}}{4k_w} \right)^{1/3} \left(\frac{\Omega}{k_{\beta}} \right)^{4/3}$	$4 \left(\frac{k_{\beta}}{k_w} \right)$	$\frac{k_{\beta}^2}{18qk_w} \left(\frac{\Omega}{k_{\beta}} \right)^4$
			$\frac{\Omega^2 L}{15k_w}^*$
L_{sat}	$4 \frac{k_w^{1/3}}{\Omega^{4/3}}$	$\frac{8k_{\beta}}{\Omega^2}$	$\frac{\Omega^2}{qk_{\beta}^2}$
$E_v <$	$\frac{\pi k_{\beta} \Omega^{4/3}}{2k_s k_{\beta}^2 k_w^{1/3}}$	$\frac{\pi \Omega^2}{4k_s k_{\beta}^2}$	$\frac{4\Omega}{k_s k_{\beta}^2}$
$E_h <$	$\frac{k_{\beta}}{2k_s k_w}$	$\frac{k_{\beta}}{2k_s k_w}$	$\frac{\Omega^3}{2k_s k_{\beta}^2 k_w}$

Characteristics of the Various Operating Regimes of the Transverse Gradient Wiggler.

* maximized with respect to q for fixed L . $\left(q = \frac{2\Omega^2}{3Lk_{\beta}^2} \right)$

weaker wiggler field. The flutter motion is thereby reduced in amplitude, the reduction in kinetic energy having gone to supply that delivered to the optical field. The two regimes differ only in their saturation mechanism. Due to the non-linear nature of the ponderomotive potential, the amplitude of the driving force at the betatron frequency decreases as the amplitude of the phase oscillation increases, and eventually vanishes. Thus the driving force is decoupled. At the same time, the underlined non-linear terms in the equations of motion cause both the betatron frequency and the driving frequency to shift as the amplitude of the oscillation grows. In the medium intensity regime the intensity is sufficiently high that the decoupling effect occurs before the detuning effect becomes effective. In the low intensity regime the growth rate is slower, and the fact that the driving force has become non resonant becomes the factor which limits growth of the wave. It should be emphasized that both regimes are non-linear. The non-linear character of the ponderomotive potential is dominant in the medium intensity regime, while the underlined non linear terms are dominant in the low intensity regime.

The high intensity regime may be thought of as a trapped particle regime analogous to the basic regime of the high extraction variable parameter wiggler (VPW) without transverse gradient. The analogy is especially close when the parameter variation consists merely of a reduction of the wiggler field intensity with z . In both regimes the particles are trapped in ponderomotive potential wells, where they execute optical phase oscillations at the optical synchrotron frequency. The gradient in wiggler field shifts the phase center of these oscillations to

one which, on the average, causes a transfer of energy to the optical field. In the TGW case the electron simultaneously drifts sidewise to a region a weaker wiggler field, while in the VPW case it convects to a position of weaker field. In both cases there is a simultaneous decrease in the flutter amplitude, thus accounting for the energy to the optical wave.

The transition between the medium and high intensity regimes requires passage through a region in which the betatron and synchrotron frequencies are equal. The numerical work of Madey and Eckstein suggests, as one might expect, rather complicated behavior, and oscillations in saturated gain may even occur. In this connection we note that the saturated gain formulas for the high and medium intensity regimes match badly at $\epsilon = 1$, $q = k_\beta$.

As contrasted to the uniform wiggler without gradient all of the TGW regimes would seem to lead to an improvement in extraction. Furthermore, as emphasized in the original proposal of the TGW, the energy spread requirements are much less severe. Extractions comparable to that of the VPW are, however, probably attainable only in the high intensity regime. Because of the similarity of the physical character of the regimes one expects that the optical intensity requirements will be similarly high. Indeed, equation (50) for the TGW may be written

$$\left(- \frac{\langle \delta \gamma \rangle}{\gamma} \right)_{\max} = .27 \frac{a_w^2}{1 + a_w^2} k_w L a_s \bar{\epsilon} < .135 k_w L a_s$$

This may be compared with

$$\left(- \frac{\langle \delta \gamma \rangle}{\gamma} \right) = .28 k_w L a_s$$

for the VPW. This latter result is based upon equation 4.8 of Ref. 2 specialized to low η_b to make it directly comparable to equation (50), with $\psi_r = .68$ rad and $f_b = .45$ as indicated by Figure 4.2 b and 4.2. ($\eta_e = \eta_i$ at low η_b) .

By way of direct comparison, Madey has considered a TGW case in which $\lambda_w = 5$ cm , $L = 20$ m , $B_w^{(nominal)} = 6$ kG , $\gamma = 354$, and $\lambda = 1\mu$. For an optical flux of 10^{12} watts/cm² he obtains numerically an extraction of 14.2%. (Our semi-analytic approach yields 12.8%). For a transverse magnet VPW case with the same parameters, except that B_w decreases linearly from 6 to 1.5 KG as one passes through the wiggler, the computed extraction (including both capture fraction and detrapping effects) is 21%.

The effective vertical focussing forces in the TGW are stronger than those in the VPW, a fact which tends to make the emittance requirements more restrictive. On the other hand, for the VPW with constant ψ_r the trapping potential (i.e., $\delta \gamma_m$) is reduced by the factor² $\Gamma(\psi_r)$. The two effects approximately cancel for the comparison example considered above.

For the VPW with transverse magnet, the horizontal emittance restriction is very weak, as there are negligible horizontal focussing

forces and negligible horizontal field variation. For convenience of beam design one takes $E_h \sim E_v$, but $E_h \gg E_v$ would presumably be achievable if the total emittance constraint were found to be excessively restrictive. In contrast, for the example being considered, the horizontal emittance requirement equation (50) is more than fifty times more restrictive than the vertical one. Thus to take full advantage of the vertical beam emittance available, a ribbon-like beam structure would be required, and it may prove very difficult to obtain such a small horizontal emittance.

Although we have not carried out the parameter exploration that would be required to reach a decisive conclusion, the above discussion provides a strong indication that compared on a steady state operation basis, the VPW will prove substantially superior to the TGW for high extraction single pass operation.

An important issue, which we have not studied, is that of the build-up of oscillations from a low level, and the nature of the optical pulse form which emerges when the system is driven by the electron micro-bunches typically produced by an RF LINAC. Qualitative changes in the operating regimes take place as build-up proceeds, and both devices are subject to detrapping by field fluctuations. It is entirely possible that important differences will emerge, and that either, neither, or both may prove desirable from this point of view.

In conclusion we wish to acknowledge that the development of the preceding theory was facilitated by the numerical work of Eckstein and Madey. Their work has also provided us with the opportunity to make some detailed quantitative and qualitative comparisons with their numerical results. The agreement is generally very good, and we have found no significant discrepancies in regions where we consider that our theory should apply.

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